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COMMENT

Size distribution with a power law in a cascade of fracture

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Abstract. A power law is found for a steady state of a cascade of fracture. A master equation of the number of fragments is presented. It is linear in the number and has a homogeneous kernel. The kernel stands for a statistical mechanism of brittle fracture on the basis of randomly distributed flaws and their spatially scale-invariant activation. It includes extended Gilvarry's theory if this could be scale invariant.

We note a phenomenon that matter is distributed in various sizes and the nature of the distribution obeys a simple power law, such as (lunar) craters (Baldwin 1964), islands (Mandelbrot 1982), earthquakes (Tsuboi 1956), ice drifts (Matsushita 1985), geometrical random clusters (Stauffer 1979), thermal critical fluctuations (Domb and Green 1972) and rock fracture (the Gaudin-Schuhmann law). They differ greatly in the size range where the law works, and from this law, a much greater variety of conjectures could be made. Thus, they appear to have nothing in common except the power law. A simple phenomenological law common to the miscellaneous phenomena would not necessarily lead to a simple mechanism. Thus, let us review briefly some investigations relevant to the law. Percolation and thermal critical phenomena have been well studied experimentally and theoretically. Critical point behaviour has been shown to be described by power laws (Kertesz *et al* 1983, Nagao 1985). In particular, in percolation it implies dilatational symmetry in clustering of particles. A number of aggregates have been studied intensively by computer simulation, experiment and theory (Herrmann 1986). Attention has been focused on their fractal features (Honda *et al* 1986). Their cluster statistics have been studied a little (Racz and Vicsek 1983, Botet and Jullien 1984, Kolb 1984, Meakin *et al* 1985). A power law is likely to be realised in gelation (van Dongen and Ernst 1985). On the other hand, fracture has been rarely studied. Only mathematical and self-symmetric procedures, assuring the expected fractal dimensionality, have been described (Mandelbrot 1982).

In this comment we present a model: a power law is realised in a steady state of a cascade of fracture, in particular brittle fracture. We may consider a stochastic process for the cascade of fracture allowing for a power law being a statistical law. Wher a mass of size x is broken by some random external forces and masses of size y corne from it, we assume the transition rate can be simplified to be a function of only x and y. Then, we have a master equation of the number $N(x, t)$ of masses of size x and at time t for a model of the cascade:

$$\partial N(x,t)/\partial t = \int_x^\infty W(y,x)N(y,t)\,dy - N(x,t)\int_0^x W(x,y)(y/x)^d\,dy \tag{1}$$

where the first integral describes the increase of the number of masses of size x by fracturing of those of size y greater than x and the second integral describes the decrease of the number of masses of size x by their fracturing into smaller ones. $W(y, x)$ represents the rate coefficient for a specific fracturing mechanism from y to x and d is the spatial dimensionality. Let us assume the kernel W is a homogeneous function of degree δ:

$$W(\lambda x, \lambda y) = \lambda^{\delta} W(x, y)$$

i.e.

$$W(x, y) = y^{\delta} W(x/y). \tag{2}$$

Equation (1) is rewritten using (2) as

$$\partial N(x, t)/\partial t = x^{\delta+1} \int_1^{\infty} W(s)\{N(xs) - s^{-(d+\delta+2)} N(x)\}\, \mathrm{d}s \tag{3}$$

where $W(s)$ is assumed positive and to work so that the integration converges.

A stationary solution of (3) is found by a power law

$$N(x, t) \propto x^{-\tau}.$$

Then

$$\tau = d + \delta + 2 \tag{4}$$

in order that the bracket $\{\ldots\}$ in (3) vanishes. To obtain such a stationary solution, we assume that there is a constant influx of new matter of some size x to the system.

Let us consider what is physically meant by the ansatz (2). We can show an instance of (2) extending Gilvarry's theory (Gilvarry 1961) of brittle fracture. His theory is as follows: microscopic flaws in a solid are activated and grown by the external forces and they finally break down the solid. The activated edge, surface and volume flaws are independently distributed with a Poisson law.

When the solid, whose volume is V_0, is instantly fractured, the number $N(x, s, v)$ of fragments falling in the (edge, surface, volume) range $(x, s, v) \sim (x+\mathrm{d}x, s+\mathrm{d}s, v+\mathrm{d}v)$ is given by

$$N(x, s, v)\, \mathrm{d}Q = (V_0/v)\, \mathrm{e}^{-Q}\, \mathrm{d}Q \tag{5}$$

where $Q(x/X, s/S, v/V) = x/X + s/S + v/V$ and X(or S, V) stands for the mean distance between the activated edge (or surface, volume) flaws. Our extension of Gilvarry's theory is that X(or S, V) should be proportional to the magnitude of the edge (or surface, volume) of the pre-fractured solid, because it is the mean distance between nucleus flaws able to break down the solid. This version has been extended to be applicable to successive fracturing, i.e. $W(y, x)$ is specified to be

$$W_g(y, x) = (y/x)^d \exp(-Q(x/y)) Q'(x/y) y^{-1}. \tag{6}$$

We apply our theory with (6) to the ice drift problem. Ice drifts are large thin plates floating on the sea surface and are supposed to be broken down by waves and by collisions with other ice drifts. Thus we obtain

$$\tau(d = 2, \delta = -1) = 3 \tag{7}$$

and the observation gives $\tau = 3.16$.

In single fracture, Gilvarry's theory can also give a power law:

$$N(x) = x^{-\tau_g(d_f, d)} \tag{8}$$

if Q could be negligible in e^{-Q} of (5), where the parameters d_f are the dimensionality of flaws and are 1, 2 or 3 according to whether they are edge, surface or volume. Then

$$\tau_g(d_f, d) \simeq d - d_f + 1$$

if $Q(z)$ can be approximated by z^{d_f}. Under these conditions, Gilvarry's theory gives

$$\tau_g(d_f, d = 2) \leq 2 \qquad \text{for ice drifts}$$

$$\tau_g(d_f = 1, d = 3) = 3 \qquad \text{for craters and earthquakes}$$

(Takeuchi and Mizutani 1968). The latter value agrees with the observed one, while the conditions assumed have not been proved.

An aggregation process also gives a cluster size distribution. Smoluchowski's coagulation model has been studied and its solution, the size distribution $N(z, t) = z^{-2}\phi(z/s(t))$, has been obtained (van Dongen and Ernst 1985) with mean cluster size $s(t)$ being an increasing function with t.

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